Division with Fractions

2.4

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A few years ago our 4-H club was making blankets to keep their lambs clean at the county fair. Each blanket required $\frac{3}{4}$ yard of material. We had 9 yards of material left over from the year before. To see how many blankets we could make, we divided 9 by $\frac{3}{4}$. The result was 12, meaning that we could make 12 lamb blankets for the fair.

Before we define division with fractions, we must first introduce the idea of *reciprocals*. Look at the following multiplication problems:

$$\frac{3}{4} \cdot \frac{4}{3} = \frac{12}{12} = 1 \qquad \frac{7}{8} \cdot \frac{8}{7} = \frac{56}{56} = 1$$

In each case the product is 1. Whenever the product of two numbers is 1, we say the two numbers are *reciprocals*.

Reciprocals

Two numbers whose product is 1 are said to be **reciprocals.** In symbols, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, because

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{a \cdot b}{b \cdot a} = \frac{a \cdot b}{a \cdot b} = 1 \quad (a \neq 0, b \neq 0)$$

Every number has a reciprocal except 0. The reason 0 does not have a reciprocal is because the product of *any* number with 0 is 0. It can never be 1. Reciprocals of whole numbers are fractions with 1 as the numerator.

For example, the reciprocal of 5 is $\frac{1}{5}$, because

$$5 \cdot \frac{1}{5} = \frac{5}{1} \cdot \frac{1}{5} = \frac{5}{5} = 1$$

Table 1 lists some numbers and their reciprocals.

Number	Reciprocal	Reason
$\frac{3}{4}$	$\frac{4}{3}$	Because $\frac{3}{4} \cdot \frac{4}{3} = \frac{12}{12} = 1$
$\frac{9}{5}$	$\frac{5}{9}$	Because $\frac{9}{5} \cdot \frac{5}{9} = \frac{45}{45} = 1$
$\frac{1}{3}$	3	Because $\frac{1}{3} \cdot 3 = \frac{1}{3} \cdot \frac{3}{1} = \frac{3}{3} = 1$
7	$\frac{1}{7}$	Because $7 \cdot \frac{1}{7} = \frac{7}{1} \cdot \frac{1}{7} = \frac{7}{7} = 1$

Table 1

The reciprocal of a negative number is also a negative number. That is,

the reciprocal of -5 is $-\frac{1}{5}$, because $-5\left(-\frac{1}{5}\right) = 1$

the reciprocal of $-\frac{3}{4}$ is $-\frac{4}{3}$, because $\left(-\frac{3}{4}\right)\left(-\frac{4}{3}\right) = 1$

Note Defining division to be the same as multiplication by the reciprocal does make sense. If we divide 6 by 2, we get 3. On the other hand, if we multiply 6 by $\frac{1}{2}$ (the reciprocal of 2), we also get 3. Whether we divide by 2 or multiply by $\frac{1}{2}$, we get the same result.

Division with fractions is accomplished by using reciprocals. More specifically, we can define division by a fraction to be the same as multiplication by its reciprocal. Here is the precise definition:

Division with Fractions

If a, b, c, and d are numbers and b, c, and d are all not equal to 0, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

This definition states that dividing by the fraction $\frac{c}{d}$ is exactly the same as multiplying by its reciprocal $\frac{d}{c}$. Because we developed the rule for multiplying fractions in Section 2.3, we do not need a new rule for division. We simply *replace the divisor by its reciprocal* and multiply. Here are some examples to illustrate the procedure.

VIDEO EXAMPLES

SECTION 2.4

Example 1 Divide $\frac{1}{2} \div \frac{1}{4}$.

Solution The divisor is $\frac{1}{4}$, and its reciprocal is $\frac{4}{1}$. Applying the definition of division for fractions, we have

$$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \cdot \frac{4}{1}$$

$$= \frac{1 \cdot 4}{2 \cdot 1}$$

$$= \frac{1 \cdot 2 \cdot 2}{2 \cdot 1}$$

$$= \frac{2}{1} = 2$$

The quotient of $\frac{1}{2}$ and $\frac{1}{4}$ is 2. Or, $\frac{1}{4}$ "goes into" $\frac{1}{2}$ two times. Logically, our definition for division of fractions seems to be giving us answers that are consistent with what we know about fractions from previous experience. Because 2 times $\frac{1}{4}$ is $\frac{2}{4}$ or $\frac{1}{2}$, it seems logical that $\frac{1}{2}$ divided by $\frac{1}{4}$ should be 2.

Example 2 Divide $\frac{3}{8} \div \frac{9}{4}$.

Solution Dividing by $\frac{9}{4}$ is the same as multiplying by its reciprocal, which is $\frac{4}{9}$:

$$\frac{3}{8} \div \frac{9}{4} = \frac{3}{8} \cdot \frac{4}{9}$$

$$= \frac{3 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3}$$

$$= \frac{1}{6}$$

The quotient of $\frac{3}{8}$ and $\frac{9}{4}$ is $\frac{1}{6}$.

Example 3 Divide $\frac{2}{3} \div 2$.

Solution The reciprocal of 2 is $\frac{1}{2}$. Applying the definition for division of fractions, we have

$$\frac{2}{3} \div 2 = \frac{2}{3} \cdot \frac{1}{2}$$

$$= \frac{2 \cdot 1}{3 \cdot 2} = \frac{1}{3}$$

Example 4 Divide $2 \div \left(-\frac{1}{3}\right)$.

Solution We replace $-\frac{1}{3}$ by its reciprocal, which is -3, and multiply:

$$2 \div \left(-\frac{1}{3}\right) = 2(-3)$$

$$= -6$$

Here are some further examples of division with fractions.

Example 5

$$-\frac{4}{27} \div \frac{16}{9} = -\frac{4}{27} \cdot \frac{9}{16}$$

$$= -\frac{4 \cdot 9}{3 \cdot 9 \cdot 4 \cdot 4}$$

$$= -\frac{1}{12}$$

In this example we did not factor the numerator and the denominator completely in order to reduce to lowest terms because, as you have probably already noticed, it is not necessary to do so. We need to factor only enough to show what numbers are common to the numerator and the denominator. If we factored completely in the second step, it would look like this:

$$= -\frac{2 \cdot 2 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = -\frac{1}{12}$$

The result is the same in both cases. From now on we will factor numerators and denominators only enough to show the factors we are dividing out.

Example 6 Divide.

a. $\frac{16}{35} \div 8$

b. $27 \div \frac{3}{2}$

Solution

a. $\frac{16}{35} \div 8 = \frac{16}{35} \cdot \frac{1}{8}$

$$= \frac{2 \cdot 8 \cdot 1}{35 \cdot 8} = \frac{2}{35}$$

b. $27 \div \frac{3}{2} = 27 \cdot \frac{2}{3}$

$$= \frac{3 \cdot 9 \cdot 2}{3} = 18$$

The next two examples combine what we have learned about division of fractions with the rule for order of operations.

Example 7 The quotient of $\frac{8}{3}$ and $\frac{1}{6}$ is increased by 5. What number results?

Solution Translating to symbols, we have

$$\frac{8}{3} \div \frac{1}{6} + 5 = \left[\frac{8}{3} \cdot \frac{6}{1}\right] + 5$$

$$= 16 + 5 = 21$$

Example 8 Simplify: $32 \div \left(\frac{4}{3}\right)^2 + 75 \div \left(\frac{5}{2}\right)^2$

Solution According to the rule for order of operations, we must first evaluate the numbers with exponents, then we divide, and finally we add.

$$\left[32 \div \left(\frac{4}{3}\right)^2\right] + \left[75 \div \left(\frac{5}{2}\right)^2\right] = 32 \div \frac{16}{9} + 75 \div \frac{25}{4}$$

$$= 32 \cdot \frac{9}{16} + 75 \cdot \frac{4}{25}$$

$$= 18 + 12$$

$$= 30$$

Applying the Concepts

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Example 9 A 4-H Club is making blankets to keep their lambs clean at the county fair. If each blanket requires $\frac{3}{4}$ yard of material, how many blankets can they make from 9 yards of material?

Solution To answer this question we must divide 9 by $\frac{3}{4}$.

$$9 \div \frac{3}{4} = 9 \cdot \frac{4}{3}$$

$$= 3 \cdot 4 = 12$$

They can make 12 blankets from the 9 yards of material.

Getting Ready for Class

After reading through the preceding section, respond in your own words and in complete sentences.

A. What do we call two numbers whose product is 1?

B. True or false? The quotient of $\frac{3}{5}$ and $\frac{3}{8}$ is the same as the product of $\frac{3}{5}$ and $\frac{3}{8}$.

C. How are multiplication and division of fractions related?

D. Dividing by $\frac{19}{9}$ is the same as multiplying by what number?

Problem Set 2.4

Find the quotient in each case by replacing the divisor by its reciprocal and multiplying.

1. $\frac{3}{4} \div \frac{1}{5}$ **2.** $\frac{1}{3} \div \frac{1}{2}$ **3.** $\frac{2}{3} \div \frac{1}{2}$ **4.** $\frac{5}{8} \div \frac{1}{4}$

5. $-6 \div \frac{2}{3}$ **6.** $-8 \div \frac{3}{4}$ **7.** $20 \div \frac{1}{10}$ **8.** $16 \div \frac{1}{8}$

9. $\frac{3}{4} \div 2$ **10.** $\frac{3}{5} \div 2$ **11.** $\frac{7}{8} \div \frac{7}{8}$ **12.** $\frac{4}{3} \div \frac{4}{3}$

13. $\frac{7}{8} \div \left(-\frac{8}{7}\right)$ **14.** $\frac{4}{3} \div \left(-\frac{3}{4}\right)$ **15.** $-\frac{9}{16} \div \left(-\frac{3}{4}\right)$ **16.** $-\frac{25}{36} \div \left(-\frac{5}{6}\right)$

15-51 odd

17. $\frac{25}{46} \div \frac{40}{69}$ **18.** $\frac{25}{24} \div \frac{15}{36}$ **19.** $\frac{13}{28} \div \frac{39}{14}$ **20.** $\frac{28}{125} \div \frac{5}{2}$

55, 57, 59

21. $\frac{27}{196} \div \frac{9}{392}$ **22.** $\frac{16}{135} \div \frac{2}{45}$ **23.** $\frac{25}{18} \div 5$ **24.** $\frac{30}{27} \div 6$

25. $6 \div \frac{4}{3}$ **26.** $12 \div \frac{4}{3}$ **27.** $-\frac{4}{3} \div 6$ **28.** $-\frac{4}{3} \div 12$

29. $\frac{3}{4} \div \frac{1}{2} \cdot 6$ **30.** $12 \div \frac{6}{7} \cdot 7$ **31.** $\frac{2}{3} \cdot \frac{3}{4} \div \frac{5}{8}$ **32.** $4 \cdot \frac{7}{6} \div 7$

33. $\frac{35}{110} \cdot \frac{80}{63} \div \frac{16}{27}$ **34.** $\frac{20}{72} \cdot \frac{42}{18} \div \frac{20}{16}$

Simplify each expression as much as possible.

35. $10 \div \left(\frac{1}{2}\right)^2$ **36.** $12 \div \left(\frac{1}{4}\right)^2$

37. $\frac{18}{35} \div \left(\frac{6}{7}\right)^2$ **38.** $\frac{48}{55} \div \left(\frac{8}{11}\right)^2$

39. $\frac{4}{5} \div \frac{1}{10} + 5$ **40.** $\frac{3}{8} \div \frac{1}{16} + 4$

41. $10 + \frac{11}{12} \div \frac{11}{24}$ **42.** $15 + \frac{13}{14} \div \frac{13}{42}$

43. $24 \div \left(\frac{2}{5}\right)^2 + 25 \div \left(\frac{5}{6}\right)^2$ **44.** $18 \div \left(\frac{3}{4}\right)^2 + 49 \div \left(\frac{7}{9}\right)^2$

45. $100 \div \left(\frac{5}{7}\right)^2 + 200 \div \left(\frac{2}{3}\right)^2$ **46.** $64 \div \left(\frac{8}{11}\right)^2 + 81 \div \left(\frac{9}{11}\right)^2$

47. What is the quotient of $\frac{3}{8}$ and $\frac{5}{8}$?

48. Find the quotient of $\frac{4}{5}$ and $\frac{16}{25}$.

49. If the quotient of 18 and $\frac{3}{5}$ is increased by 10, what number results?

50. If the quotient of 50 and $\frac{5}{3}$ is increased by 8, what number results?

51. Show that multiplying 3 by 5 is the same as dividing 3 by $\frac{1}{5}$.

52. Show that multiplying 8 by $\frac{1}{2}$ is the same as dividing 8 by 2.

Applying the Concepts

Although many of the application problems that follow involve division with fractions, some do not. Be sure to read the problems carefully.

53. Sewing If $\frac{6}{7}$ yard of material is needed to make a blanket, how many blankets can be made from 12 yards of material?

54. Manufacturing A clothing manufacturer is making scarves that require $\frac{3}{8}$ yard of material each. How many can be made from 27 yards of material?

55. Capacity Suppose a bag of candy holds exactly $\frac{1}{4}$ pound of candy. How many of these bags can be filled from 12 pounds of candy?

56. Capacity A certain size bottle holds exactly $\frac{4}{5}$ pint of liquid. How many of these bottles can be filled from a 20-pint container?

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57. Cooking A man is making cookies from a recipe that calls for $\frac{3}{4}$ teaspoon of oil. If the only measuring spoon he can find is a $\frac{1}{8}$ teaspoon, how many of these will he have to fill with oil in order to have a total of $\frac{3}{4}$ teaspoon of oil?

58. Cooking A cake recipe calls for $\frac{1}{2}$ cup of sugar. If the only measuring cup available is a $\frac{1}{8}$ cup, how many of these will have to be filled with sugar to make a total of $\frac{1}{2}$ cup of sugar?

59. Student Population If 14 of every 32 students attending Cuesta College are female, what fraction of the students is female? (Simplify your answer.)

60. Population If 27 of every 48 residents of a small town are male, what fraction of the population is male? (Simplify your answer.)

Getting Ready for the Next Section

Write each fraction as an equivalent fraction with denominator 6.

61. $\frac{1}{2}$ **62.** $\frac{3}{1}$ **63.** $\frac{3}{2}$ **64.** $\frac{2}{3}$

Write each fraction as an equivalent fraction with denominator 12.

65. $\frac{1}{6}$ **66.** $\frac{1}{2}$ **67.** $\frac{2}{3}$ **68.** $\frac{3}{4}$

Write each fraction as an equivalent fraction with denominator 30.

69. $\frac{7}{15}$ **70.** $\frac{3}{10}$ **71.** $\frac{3}{5}$ **72.** $\frac{1}{6}$

Write each fraction as an equivalent fraction with denominator 24.

73. $\frac{1}{2}$ **74.** $\frac{1}{4}$ **75.** $\frac{1}{6}$ **76.** $\frac{1}{8}$

Write each fraction as an equivalent fraction with denominator 36.

77. $\frac{5}{12}$ **78.** $\frac{7}{18}$