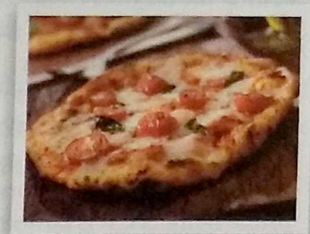
Prime Numbers, Factors, and Reducing to Lowest Terms

2.2

Suppose you and a friend decide to split a medium-sized pizza for lunch. When the pizza is delivered you find that it has been cut into eight equal pieces. If you eat four pieces, you have eaten $\frac{4}{8}$ of the pizza, but you also know that you have eaten $\frac{1}{2}$ of the pizza. The fraction $\frac{4}{8}$ is equivalent to the fraction $\frac{1}{2}$; that is, they both have the same value. The mathematical process we use to rewrite $\frac{4}{8}$ as $\frac{1}{2}$ is called *reducing to lowest terms.* Before we look at that process, we need to define some new terms. Here is our first one.



> **Prime Number**
> A **prime number** is any whole number greater than 1 that has exactly two divisors—itself and 1. (A number is a divisor of another number if it divides it without a remainder.)

$$\text{Prime numbers} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \dots\}$$

The list goes on indefinitely. Each number in the list has exactly two distinct divisors—itself and 1.

> **Composite Number**
> Any whole number greater than 1 that is not a prime number is called a **composite number.** A composite number always has at least one divisor other than itself and 1.

VIDEO EXAMPLES

SECTION 2.2

Example 1 Identify each of the numbers below as either a prime number or a composite number. For those that are composite, give two divisors other than the number itself or 1.

a. 43 **b.** 12

Solution

a. 43 is a prime number, because the only numbers that divide it without a remainder are 43 and 1.

b. 12 is a composite number, because it can be written as

$$12 = 4 \cdot 3$$

which means that 4 and 3 are divisors of 12. (These are not the only divisors of 12; other divisors are 1, 2, 6, and 12.)

> *Note* You may have already noticed that the word *divisor* as we are using it here means the same as the word *factor.* A divisor and a factor of a number are the same thing. A number can't be a divisor of another number without also being a factor of it.

Every composite number can be written as the product of prime factors. Let's look at the composite number 108. We know we can write 108 as $2 \cdot 54$. The number 2 is a prime number, but 54 is not prime. Because 54 can be written as $2 \cdot 27$, we have

$$\begin{aligned} 108 &= 2 \cdot 54 \\ &= 2 \cdot 2 \cdot 27 \end{aligned}$$

Now the number 27 can be written as $3 \cdot 9$ or $3 \cdot 3 \cdot 3$ (because $9 = 3 \cdot 3$), so

$$108 = 2 \cdot 54$$
$$108 = 2 \cdot 2 \cdot 27$$
$$108 = 2 \cdot 2 \cdot 3 \cdot 9$$
$$108 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$$

> ***Note*** This process works by writing the original composite number as the product of any two of its factors and then writing any factor that is not prime as the product of any two of its factors. The process is continued until all factors are prime numbers. It is customary to finish by using exponents where appropriate and writing the factors in increasing order; that is, from smallest to largest.

This last line is the number 108 written as the product of prime factors. We can use exponents to rewrite the last line:

$$108 = 2^2 \cdot 3^3$$

Example 2 Factor 60 into a product of prime factors.

Solution We begin by writing 60 as $6 \cdot 10$ and continue factoring until all factors are prime numbers:

$$60 = 6 \cdot 10$$
$$= 2 \cdot 3 \cdot 2 \cdot 5 \quad \text{Use exponents and write factors from smallest to largest}$$
$$= 2^2 \cdot 3 \cdot 5$$

Notice that if we had started by writing 60 as $3 \cdot 20$, we would have achieved the same result:

$$60 = 3 \cdot 20$$
$$= 3 \cdot 2 \cdot 10$$
$$= 3 \cdot 2 \cdot 2 \cdot 5 \quad \text{Use exponents and write in increasing order}$$
$$= 2^2 \cdot 3 \cdot 5$$

> ***Note on Divisibility***
> There are some "shortcuts" to finding the divisors of a number. For instance, if a number ends in 0 or 5, then it is divisible by 5. If a number ends in an even number (0, 2, 4, 6, or 8), then it is divisible by 2. A number is divisible by 3 if the sum of its digits is divisible by 3. For example, 921 is divisible by 3 because the sum of its digits is $9 + 2 + 1 = 12$, which is divisible by 3.

We can use the method of factoring numbers into prime factors to help reduce fractions to lowest terms. Here is the definition for lowest terms.

> **Lowest Terms**
> A fraction is said to be in **lowest terms** if the numerator and the denominator have no factors in common other than the number 1.

Clarification 1 The fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, and $\frac{4}{5}$ are all in lowest terms, because in each case the numerator and the denominator have no factors other than 1 in common. That is, in each fraction, no number other than 1 divides both the numerator and the denominator exactly (without a remainder).

Clarification 2 The fraction $\frac{6}{8}$ is not written in lowest terms, because the numerator and the denominator are both divisible by 2. To write $\frac{6}{8}$ in lowest terms, we apply Property 2 from Section 2.1 and divide both the numerator and the denominator by 2:

$$\frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$$

The fraction $\frac{3}{4}$ is in lowest terms, because 3 and 4 have no factors in common except the number 1.

Reducing a fraction to lowest terms is simply a matter of dividing the numerator and the denominator by all the factors they have in common. We know from Property 2 of Section 2.1 that this will produce an equivalent fraction.

Example 3 Reduce the fraction $\frac{12}{15}$ to lowest terms by first factoring the numerator and the denominator into prime factors and then dividing both the numerator and the denominator by the factor they have in common.

Solution The numerator and the denominator factor as follows:

$$12 = 2 \cdot 2 \cdot 3 \quad \text{and} \quad 15 = 3 \cdot 5$$

The factor they have in common is 3. Property 2 tells us that we can divide both terms of a fraction by 3 to produce an equivalent fraction. So

$$\frac{12}{15} = \frac{2 \cdot 2 \cdot 3}{3 \cdot 5} \qquad \text{Factor the numerator and the denominator completely}$$

$$= \frac{2 \cdot 2 \cdot 3 \div 3}{3 \cdot 5 \div 3} \qquad \text{Divide by 3}$$

$$= \frac{2 \cdot 2}{5} = \frac{4}{5}$$

The fraction $\frac{4}{5}$ is equivalent to $\frac{12}{15}$ and is in lowest terms, because the numerator and the denominator have no factors other than 1 in common.

We can shorten the work involved in reducing fractions to lowest terms by using a slash to indicate division. For example, we can write the above problem this way:

$$\frac{12}{15} = \frac{2 \cdot 2 \cdot \cancel{3}}{\cancel{3} \cdot 5} = \frac{4}{5}$$

So long as we understand that the slashes through the 3's indicate that we have divided both the numerator and the denominator by 3, we can use this notation.

Applying the Concepts

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Example 4 Laura is having a party. She puts 4 six-packs of diet soda in a cooler for her guests. At the end of the party she finds that only 4 sodas have been consumed. What fraction of the sodas are left? Write your answer in lowest terms.

Solution She had 4 six-packs of soda, which is $4(6) = 24$ sodas. Only 4 were consumed at the party, so 20 are left. The fraction of sodas left is

$$\frac{20}{24}$$

Factoring 20 and 24 completely and then dividing out both the factors they have in common gives us

$$\frac{20}{24} = \frac{\cancel{2} \cdot \cancel{2} \cdot 5}{\cancel{2} \cdot \cancel{2} \cdot 2 \cdot 3} = \frac{5}{6}$$

Note The slashes in Example 4 indicate that we have divided both the numerator and the denominator by $2 \cdot 2$, which is equal to 4. With some fractions it is apparent at the start what number divides the numerator and the denominator. For instance, you may have recognized that both 20 and 24 in Example 4 are divisible by 4. We can divide both terms by 4 without factoring first, just as we did in Section 2.1. Property 2 guarantees that dividing both terms of a fraction by 4 will produce an equivalent fraction:

$$\frac{20}{24} = \frac{20 \div 4}{24 \div 4} = \frac{5}{6}$$

Example 5 Reduce $\frac{6}{42}$ to lowest terms.

Solution We begin by factoring both terms. We then divide through by any factors common to both terms:

$$\frac{6}{42} = \frac{\cancel{2} \cdot \cancel{3}}{\cancel{2} \cdot \cancel{3} \cdot 7} = \frac{1}{7}$$

We must be careful in a problem like this to remember that the slashes indicate division. They are used to indicate that we have divided both the numerator and the denominator by $2 \cdot 3 = 6$. The result of dividing the numerator 6 by $2 \cdot 3$ is 1. It is a very common mistake to call the numerator 0 instead of 1 or to leave the numerator out of the answer.

Example 6 Reduce $\frac{4}{40}$ to lowest terms.

$$\frac{4}{40} = \frac{\cancel{2} \cdot \cancel{2} \cdot 1}{\cancel{2} \cdot \cancel{2} \cdot 2 \cdot 5} = \frac{1}{10}$$

Example 7 Reduce $-\frac{105}{30}$ to lowest terms.

$$-\frac{105}{30} = -\frac{\cancel{3} \cdot \cancel{5} \cdot 7}{2 \cdot \cancel{3} \cdot \cancel{5}} = -\frac{7}{2}$$

Getting Ready for Class

After reading through the preceding section, respond in your own words and in complete sentences.

A. What is a prime number?

B. Why is the number 22 a composite number?

C. Factor 120 into a product of prime factors.

D. What is meant by the phrase "a fraction in lowest possible terms"?

Problem Set 2.2

Identify each of the numbers below as either a prime number or a composite number. For those that are composite, give at least one divisor (factor) other than the number itself or the number 1.

1. 11 **2.** 23 **3.** 105 **4.** 41

5. 81 **6.** 50 **7.** 13 **8.** 219

Factor each of the following into a product of prime factors.

9. 12 **10.** 8 **11.** 81 **12.** 210

13. 215 **14.** 75 **15.** 15 **16.** 42

Reduce each fraction to lowest terms.

17. $\frac{5}{10}$ **18.** $\frac{3}{6}$ **19.** $\frac{4}{6}$ **20.** $\frac{4}{10}$

21. $-\frac{8}{10}$ **22.** $-\frac{6}{10}$ **23.** $\frac{36}{20}$ **24.** $\frac{32}{12}$

25. $\frac{42}{66}$ **26.** $\frac{36}{60}$ **27.** $\frac{24}{40}$ **28.** $\frac{50}{75}$

29. $\frac{14}{98}$ **30.** $\frac{12}{84}$ **31.** $\frac{70}{90}$ **32.** $\frac{80}{90}$

33. $\frac{42}{30}$ **34.** $\frac{60}{36}$ **35.** $-\frac{18}{90}$ **36.** $-\frac{150}{210}$

37. $\frac{110}{70}$ **38.** $\frac{45}{75}$ **39.** $\frac{180}{108}$ **40.** $\frac{105}{30}$

41. $\frac{96}{108}$ **42.** $\frac{66}{84}$ **43.** $\frac{126}{165}$ **44.** $\frac{210}{462}$

45. $\frac{102}{114}$ **46.** $\frac{255}{285}$ **47.** $\frac{294}{693}$ **48.** $\frac{273}{385}$

49. Reduce each fraction to lowest terms.

a. $\frac{6}{51}$ **b.** $\frac{6}{52}$ **c.** $\frac{6}{54}$ **d.** $\frac{6}{56}$ **e.** $\frac{6}{57}$

50. Reduce each fraction to lowest terms.

a. $\frac{6}{42}$ **b.** $\frac{6}{44}$ **c.** $\frac{6}{45}$ **d.** $\frac{6}{46}$ **e.** $\frac{6}{48}$

51. Reduce each fraction to lowest terms.

a. $\frac{2}{90}$ **b.** $\frac{3}{90}$ **c.** $\frac{5}{90}$ **d.** $\frac{6}{90}$ **e.** $\frac{9}{90}$

52. Reduce each fraction to lowest terms.

a. $\frac{3}{105}$ **b.** $\frac{5}{105}$ **c.** $\frac{7}{105}$ **d.** $\frac{15}{105}$ **e.** $\frac{21}{105}$

53. The answer to each problem below is wrong. Give the correct answer.

a. $\frac{5}{15} = \frac{5}{3 \cdot 5} = \frac{0}{3}$ **b.** $\frac{5}{6} = \frac{3 + 2}{4 + 2} = \frac{3}{4}$ **c.** $\frac{6}{30} = \frac{2 \cdot 3}{2 \cdot 3 \cdot 5} = 5$

54. The answer to each problem below is wrong. Give the correct answer.

a. $\frac{10}{20} = \frac{7 + 3}{17 + 3} = \frac{7}{17}$ **b.** $\frac{9}{36} = \frac{3 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 3} = \frac{0}{4}$ **c.** $\frac{4}{12} = \frac{2 \cdot 2}{2 \cdot 2 \cdot 3} = 3$

55. Which of the fractions $\frac{6}{8}$, $\frac{15}{20}$, $\frac{9}{16}$, and $\frac{21}{28}$ does not reduce to $\frac{3}{4}$?

56. Which of the fractions $\frac{4}{9}$, $\frac{10}{15}$, $\frac{8}{12}$, and $\frac{6}{12}$ do not reduce to $\frac{2}{3}$?

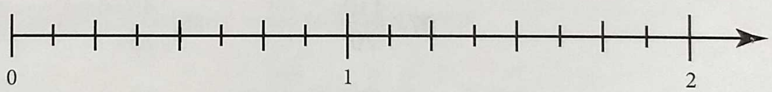
The number line below extends from 0 to 2, with the segment from 0 to 1 and the segment from 1 to 2 each divided into 8 equal parts. Locate each of the following numbers on this number line.

57. $\frac{1}{2}, \frac{2}{4}, \frac{4}{8}$, and $\frac{8}{16}$

58. $\frac{3}{2}, \frac{6}{4}, \frac{12}{8}$, and $\frac{24}{16}$

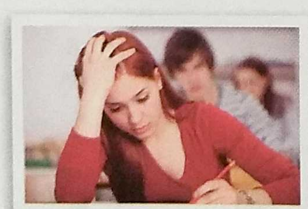
59. $\frac{5}{4}, \frac{10}{8}$, and $\frac{20}{16}$

60. $\frac{1}{4}, \frac{2}{8}$, and $\frac{4}{16}$



Applying the Concepts

61. Income A family's monthly income is $2,400, and they spend $600 each month on food. Write the amount they spend on food as a fraction of their monthly income in lowest terms.



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62. Hours and Minutes There are 60 minutes in 1 hour. What fraction of an hour is 20 minutes? Write your answer in lowest terms.

63. Final Exam Suppose 33 people took the final exam in a math class. If 11 people got an A on the final exam, what fraction of the students did not get an A on the exam? Write your answer in lowest terms.

64. Income Tax A person making $21,000 a year pays $3,000 in income tax. What fraction of the person's income is paid as income tax? Write your answer in lowest terms.

Nutrition The nutrition labels below are from two different granola bars. Use them to work Problems 65 – 70.

Granola bar 1

Nutrition Facts
Serving Size 2 bars (47g)
Servings Per Container: 6

Amount Per Serving	
Calories 210	Calories from fat 70
	% Daily Value*
Total Fat 8g	12%
Saturated Fat 1g	5%
Cholesterol 0mg	0%
Sodium 150mg	6%
Total Carbohydrate 32g	11%
Fiber 2g	10%
Sugars 12g	
Protein 4g	

*Percent Daily Values are based on a 2,000 calorie diet. Your daily values may be higher or lower depending on your calorie needs.

Granola bar 2

Nutrition Facts
Serving Size 1 bar (21g)
Servings Per Container: 8

Amount Per Serving	
Calories 80	Calories from fat 15
	% Daily Value*
Total Fat 1.5g	2%
Saturated Fat 0g	0%
Cholesterol 0mg	0%
Sodium 60mg	3%
Total Carbohydrate 16g	5%
Fiber 1g	4%
Sugars 5g	
Protein 2g	

*Percent Daily Values are based on a 2,000 calorie diet. Your daily values may be higher or lower depending on your calorie needs.

65. What fraction of the calories in granola bar 1 comes from fat?

66. What fraction of the calories in granola bar 2 comes from fat?

67. For granola bar 1, what fraction of the total fat is from saturated fat?

68. For granola bar 2, what fraction of the total fat is from saturated fat?

69. What fraction of the total carbohydrates in granola bar 1 is from sugar?

70. What fraction of the total carbohydrates in granola bar 2 is from sugar?

Getting Ready for the Next Section

Multiply.

71. $1 \cdot 3 \cdot 1$

72. $2 \cdot 4 \cdot 5$

73. $2(-3)$

74. $(-7)(-5)$

75. $5 \cdot 5 \cdot 1$

76. $6 \cdot 6 \cdot 2$

Factor into prime factors.

77. 60

78. 72

79. $15 \cdot 4$

80. $8 \cdot 9$

Expand and multiply.

81. 3^2

82. 4^2

83. 5^2

84. 6^2