Fractions and Mixed Numbers

2

Chapter Outline

If you have had any problems with or testing of your thyroid gland, then you may have come in contact with radioactive Iodine-131. Like all radioactive elements, Iodine-131 decays naturally. The half-life of Iodine-131 is 8 days, which means that every 8 days a sample of Iodine-131 will decrease to half its original amount. The table and accompanying line graph illustrate the radioactive decay of Iodine-131. As we saw in Chapter 1, line graphs give us a way of taking the information in a table and displaying it in a more visual form.

There are many radioactive materials in the world we inhabit. In each case, the simple fractions shown here are a straightforward, simple way to describe the way in which these materials decay.

Iodine-131 Decay

Days Since Ingestion	Fraction of Dose Remaining
0	1
8	$\frac{1}{2}$
16	$\frac{1}{4}$
24	$\frac{1}{8}$
32	$\frac{1}{16}$
40	$\frac{1}{32}$

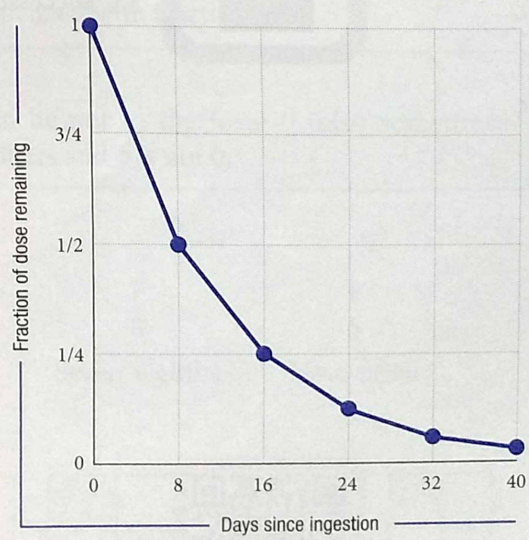


Figure 1

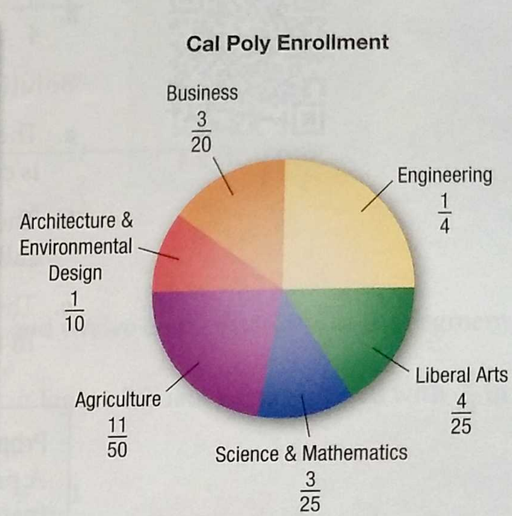
The Meaning and Properties of Fractions

2.1

The information in the table below was taken from the website for Cal Poly State University in California. The pie chart was created from the table. Both the table and pie chart use fractions to specify how the students at Cal Poly are distributed among the different schools within the university.

CAL POLY STATE UNIVERSITY ENROLLMENT

School	Fraction of Students
Agriculture	$\frac{11}{50}$
Architecture and Environmental Design	$\frac{1}{10}$
Business	$\frac{3}{20}$
Engineering	$\frac{1}{4}$
Liberal Arts	$\frac{4}{25}$
Science and Mathematics	$\frac{3}{25}$



From the table, we see that $\frac{1}{4}$ (one-fourth) of the students are enrolled in the School of Engineering. This means that one out of every four students at Cal Poly is studying Engineering. The fraction $\frac{1}{4}$ tells us we have 1 part of 4 equal parts. That is, the students at Cal Poly could be divided into 4 equal groups, so that one of the groups contained all the engineering students and only engineering students.

Figure 1 at the left shows a rectangle that has been divided into equal parts, four different ways. The shaded area for each rectangle is $\frac{1}{2}$ the total area.

Now that we have an intuitive idea of the meaning of fractions, here are the more formal definitions and vocabulary associated with fractions.

Note As we mentioned in Chapter R, when we use a letter to represent a number, or a group of numbers, that letter is called a variable. In the definition here, we are restricting the numbers that the variable b can represent to numbers other than 0. As you will see later in the chapter, we do this to avoid writing an expression that would imply division by the number 0.

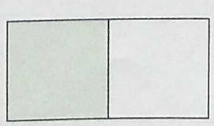
Fraction

A **fraction** is any number that can be put in the form $\frac{a}{b}$ (also sometimes written a/b), where a and b are numbers and b is not 0.

Some examples of fractions are:

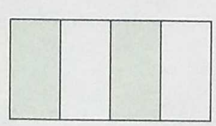
$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{9}{5}$
One-half	Three-fourths	Seven-eighths	Nine-fifths

Four Ways to Visualize $\frac{1}{2}$



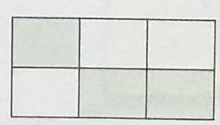
$\frac{1}{2}$ is shaded

Figure 1a



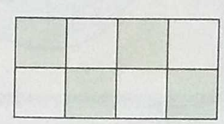
$\frac{2}{4}$ are shaded

Figure 1b



$\frac{3}{6}$ are shaded

Figure 1c



$\frac{4}{8}$ are shaded

Figure 1d

Numerator and Denominator

For the fraction $\frac{a}{b}$, a and b are called the **terms** of the fraction. More specifically, a is called the **numerator,** and b is called the **denominator.**

VIDEO EXAMPLES

SECTION 2.1

Example 1 Name the numerator and denominator for each fraction.

a. $\frac{3}{4}$ **b.** $\frac{a}{5}$ **c.** $\frac{7}{1}$

Solution In each case we use the definition above.

a. The terms of the fraction $\frac{3}{4}$ are 3 and 4. The 3 is called the numerator, and the 4 is called the denominator.

b. The numerator of the fraction $\frac{a}{5}$ is a. The denominator is 5. Both a and 5 are called terms.

c. The number 7 may also be put in fraction form, because it can be written as $\frac{7}{1}$. In this case, 7 is the numerator and 1 is the denominator.

Proper and Improper Fractions

A **proper fraction** is a fraction in which the numerator is less than the denominator. If the numerator is greater than or equal to the denominator, the fraction is called an **improper fraction.**

Clarification 1 The fractions $\frac{3}{4}$, $\frac{1}{8}$, and $\frac{9}{10}$ are all proper fractions, because in each case the numerator is less than the denominator.

Clarification 2 The numbers $\frac{9}{5}$, $\frac{10}{10}$, and 6 are all improper fractions, because in each case the numerator is greater than or equal to the denominator. (Remember that 6 can be written as $\frac{6}{1}$, in which case 6 is the numerator and 1 is the denominator.)

Fractions on the Number Line

We can give meaning to the fraction $\frac{2}{3}$ by using a number line. If we take that part of the number line from 0 to 1 and divide it into *three equal parts,* we say that we have divided it into *thirds* (see Figure 2). Each of the three segments is $\frac{1}{3}$ (one third) of the whole segment from 0 to 1.

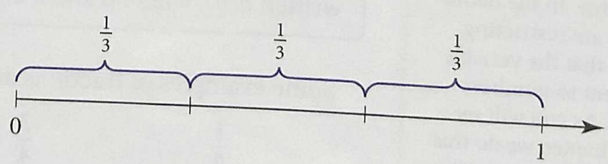
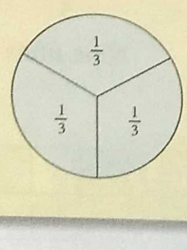


Figure 2

Note There are many ways to give meaning to fractions like $\frac{2}{3}$ other than by using the number line. One popular way is to think of cutting a pie into three equal pieces, as shown below. If you take two of the pieces, you have taken $\frac{2}{3}$ of the pie.



Two of these smaller segments together are $\frac{2}{3}$ (two thirds) of the whole segment. And three of them would be $\frac{3}{3}$ (three thirds), or the whole segment, as indicated in Figure 3.

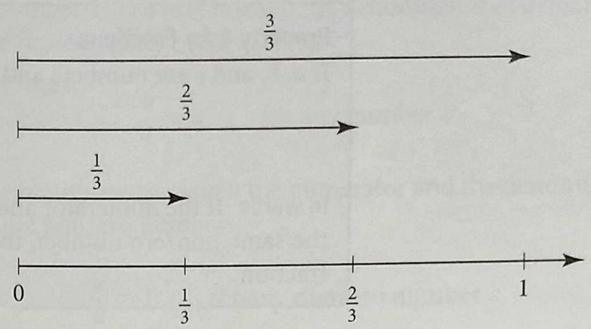


Figure 3

Let's do the same thing again with six and twelve equal divisions of the segment from 0 to 1 (see Figure 4).

The same point that we labeled with $\frac{1}{3}$ in Figure 3 is labeled with $\frac{2}{6}$ and with $\frac{4}{12}$ in Figure 4. It must be true then that

$$\frac{4}{12} = \frac{2}{6} = \frac{1}{3}$$

Although these three fractions look different, each names the same point on the number line, as shown in Figure 4. All three fractions have the same *value*, because they all represent the same number.

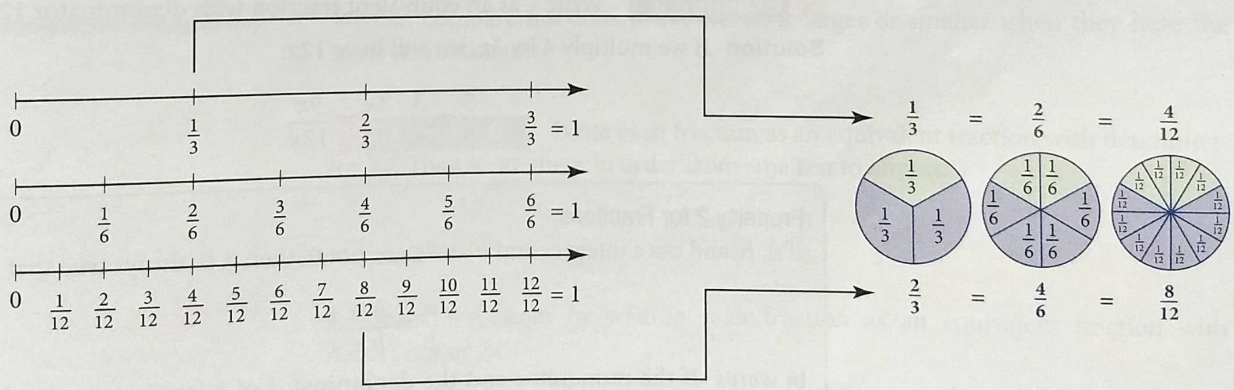


Figure 4

Equivalent Fractions

Fractions that represent the same number are said to be **equivalent.** Equivalent fractions may look different, but they must have the same value.

It is apparent that every fraction has many different representations, each of which is equivalent to the original fraction. The next two properties give us a way of changing the terms of a fraction without changing its value.

Property 1 for Fractions

If a, b, and c are numbers and b and c are not 0, then it is always true that

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$$

In words If the numerator and the denominator of a fraction are multiplied by the same nonzero number, the resulting fraction is equivalent to the original fraction.

Example 2 Write $\frac{3}{4}$ as an equivalent fraction with denominator 20.

Solution The denominator of the original fraction is 4. The fraction we are trying to find must have a denominator of 20. We know that if we multiply 4 by 5, we get 20. Property 1 indicates that we are free to multiply the denominator by 5 so long as we do the same to the numerator.

$$\frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5} = \frac{15}{20}$$

The fraction $\frac{15}{20}$ is equivalent to the fraction $\frac{3}{4}$.

Example 3 Write $\frac{3}{4}$ as an equivalent fraction with denominator $12x$.

Solution If we multiply 4 by $3x$, we will have $12x$:

$$\frac{3}{4} = \frac{3 \cdot 3x}{4 \cdot 3x} = \frac{9x}{12x}$$

Property 2 for Fractions

If a, b, and c are integers and b and c are not 0, then it is always true that

$$\frac{a}{b} = \frac{a \div c}{b \div c}$$

In words If the numerator and the denominator of a fraction are divided by the same nonzero number, the resulting fraction is equivalent to the original fraction.

Example 4 Write $\frac{10}{12}$ as an equivalent fraction with denominator 6.

Solution If we divide the original denominator 12 by 2, we obtain 6. Property 2 indicates that if we divide both the numerator and the denominator by 2, the resulting fraction will be equal to the original fraction:

$$\frac{10}{12} = \frac{10 \div 2}{12 \div 2} = \frac{5}{6}$$

The Number 1 and Fractions

There are two situations that occur frequently in mathematics which involve fractions and the number 1. The first is when the denominator of a fraction is 1. In this case, if we let a represent any number, then

$$\frac{a}{1} = a \qquad \text{for any number } a$$

The second situation occurs when the numerator and the denominator of a fraction are the same nonzero number:

$$\frac{a}{a} = 1 \qquad \text{for any nonzero number } a$$

Note Since the fraction bar is a way of signifying division, we can simplify the expression in Example 5c by dividing 48 by 24. We could also transform the expression into the form $\frac{a}{1}$ by using Property 2 for Fractions and dividing the numerator and denominator by 24. This leaves us with $\frac{2}{1}$, or 2, since $\frac{a}{1} = a$. Either way gives us the same result.

Example 5 Simplify each expression.

a. $\frac{24}{1}$ **b.** $\frac{24}{24}$ **c.** $\frac{48}{24}$ **d.** $\frac{72}{24}$

Solution In each case we divide the numerator by the denominator:

a. $\frac{24}{1} = 24$ **b.** $\frac{24}{24} = 1$ **c.** $\frac{48}{24} = 2$ **d.** $\frac{72}{24} = 3$

Comparing Fractions

We can compare fractions to see which is larger or smaller when they have the same denominator.

Example 6 Write each fraction as an equivalent fraction with denominator 24. Then write them in order from smallest to largest.

$$\frac{5}{8} \quad \frac{5}{6} \quad \frac{3}{4} \quad \frac{2}{3}$$

Solution We begin by writing each fraction as an equivalent fraction with denominator 24.

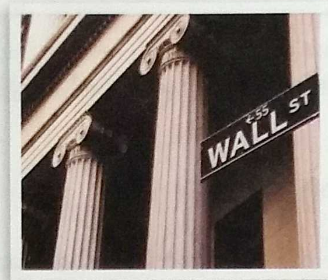
$$\frac{5}{8} = \frac{15}{24} \qquad \frac{5}{6} = \frac{20}{24} \qquad \frac{3}{4} = \frac{18}{24} \qquad \frac{2}{3} = \frac{16}{24}$$

Now that they all have the same denominator, the smallest fraction is the one with the smallest numerator and the largest fraction is the one with the largest numerator. Writing them in order from smallest to largest we have:

$$\frac{15}{24} < \frac{16}{24} < \frac{18}{24} < \frac{20}{24}$$

or

$$\frac{5}{8} < \frac{2}{3} < \frac{3}{4} < \frac{5}{6}$$



Descriptive Statistics Scatter Diagrams and Line Graphs

The table and bar chart give the daily gain in the price of eCollege.com stock for one week, when stock prices were given in terms of fractions instead of decimals.

Change in Stock Price

Day	Gain
Monday	$\frac{3}{4}$
Tuesday	$\frac{9}{16}$
Wednesday	$\frac{3}{32}$
Thursday	$\frac{7}{32}$
Friday	$\frac{1}{16}$

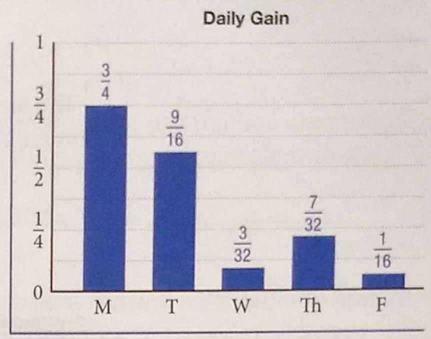


Figure 5 Bar Chart

Figure 6 below shows another way to visualize the information in the table. Recall from Chapter 1 that in a *scatter diagram*, dots are used instead of the bars shown in Figure 5 to represent the gain in stock price for each day of the week. If we connect the dots in Figure 6 with straight lines, we produce the *line graph* in Figure 7.

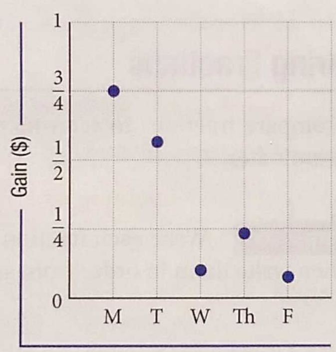


Figure 6 Scatter Diagram

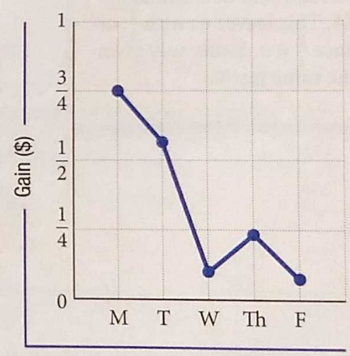


Figure 7 Line Graph

Getting Ready for Class

After reading through the preceding section, respond in your own words and in complete sentences.

A. Explain what a fraction is.
B. Which term in the fraction $\frac{7}{8}$ is the numerator?
C. Is the fraction $\frac{3}{9}$ a proper fraction?
D. What word do we use to describe fractions such as $\frac{1}{5}$ and $\frac{4}{20}$, which look different, but have the same value?

Problem Set 2.1

Name the numerator of each fraction.

1. $\frac{1}{3}$ **2.** $\frac{1}{4}$ **3.** $\frac{2}{3}$ **4.** $\frac{2}{4}$ **5.** $\frac{x}{8}$ **6.** $\frac{y}{10}$ **7.** $\frac{a}{b}$ **8.** $\frac{x}{y}$

Name the denominator of each fraction.

9. $\frac{2}{5}$ **10.** $\frac{3}{5}$ **11.** 6 **12.** 2 **13.** $\frac{a}{12}$ **14.** $\frac{b}{14}$

Complete the tables.

15.

Numerator	Denominator	Fraction
a	b	$\frac{a}{b}$
3	5	
1		$\frac{1}{7}$
	y	$\frac{x}{y}$
$x + 1$	x	

16.

Numerator	Denominator	Fraction
a	b	$\frac{a}{b}$
2	9	
	3	$\frac{4}{3}$
1		$\frac{1}{x}$
x		$\frac{x}{x+1}$

17. For the set of numbers $\left\{\frac{3}{4}, \frac{6}{5}, \frac{12}{3}, \frac{1}{2}, \frac{9}{10}, \frac{20}{10}\right\}$, list all the proper fractions.

18. For the set of numbers $\left\{\frac{1}{8}, \frac{7}{9}, \frac{6}{3}, \frac{18}{6}, \frac{3}{5}, \frac{9}{8}\right\}$, list all the improper fractions.

Indicate whether each of the following is *True* or *False*.

19. Every whole number greater than 1 can also be expressed as an improper fraction.

20. Some improper fractions are also proper fractions.

21. Adding the same number to the numerator and the denominator of a fraction will not change its value.

22. The fractions $\frac{3}{4}$ and $\frac{9}{16}$ are equivalent.

Divide the numerator and the denominator of each of the following fractions by 2.

23. $\frac{6}{8}$ **24.** $\frac{10}{12}$ **25.** $\frac{86}{94}$ **26.** $\frac{106}{142}$

Divide the numerator and the denominator of each of the following fractions by 3.

27. $\frac{12}{9}$ **28.** $\frac{33}{27}$ **29.** $\frac{39}{51}$ **30.** $\frac{57}{69}$

Write each of the following fractions as an equivalent fraction with denominator 6.

31. $\frac{2}{3}$ **32.** $\frac{1}{2}$ **33.** $\frac{55}{66}$ **34.** $\frac{65}{78}$

Write each of the following fractions as an equivalent fraction with denominator 12.

35. $\frac{2}{3}$ **36.** $\frac{5}{6}$ **37.** $\frac{56}{84}$ **38.** $\frac{143}{156}$

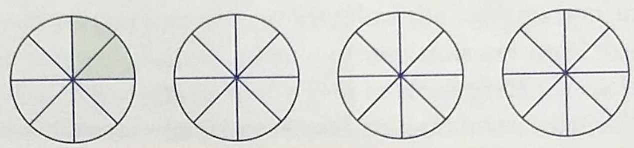
Write each fraction with denominator $12x$.

39. $\frac{1}{6}$ **40.** $\frac{3}{4}$

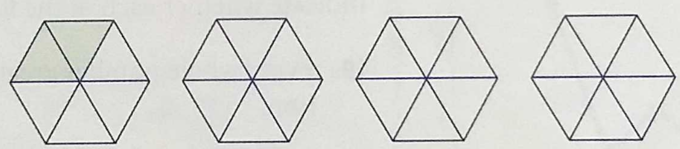
Write each number as an equivalent fraction with denominator 8.

41. 2 **42.** 1 **43.** 5 **44.** 8

45. One-fourth of the first circle below is shaded. Use the other three circles to show three other ways to shade one-fourth of the circle.



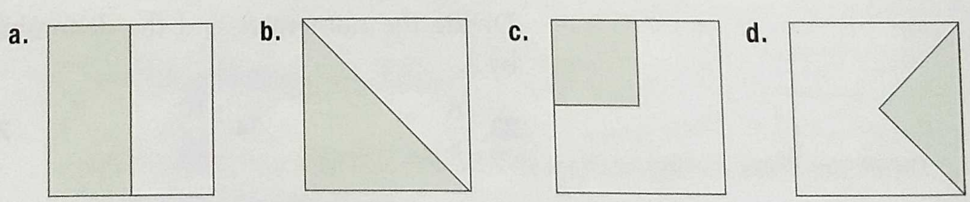
46. The six-sided figures below are hexagons. One-third of the first hexagon is shaded. Shade the other three hexagons to show three other ways to represent one-third.



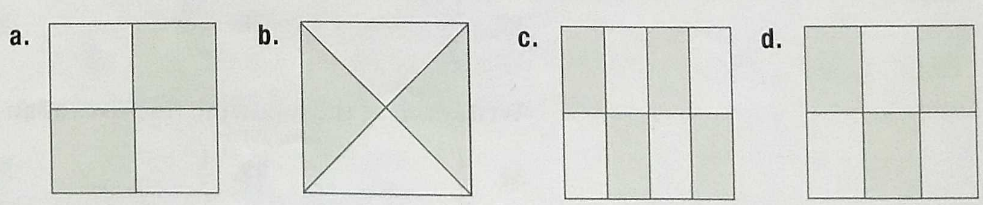
Simplify by dividing the numerator by the denominator.

47. $\frac{3}{1}$ **48.** $\frac{3}{3}$ **49.** $\frac{6}{3}$ **50.** $\frac{12}{3}$ **51.** $\frac{37}{1}$ **52.** $\frac{37}{37}$

53. For each square below, what fraction of the area is given by the shaded region?



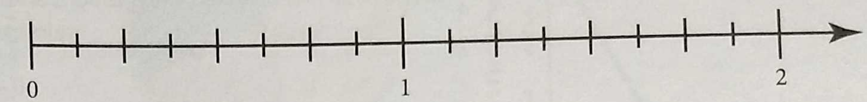
54. For each square below, what fraction of the area is given by the shaded region?



The number line below extends from 0 to 2, with the segment from 0 to 1 and the segment from 1 to 2 each divided into 8 equal parts. Locate each of the following numbers on this number line.

55. $\frac{1}{4}$ **56.** $\frac{1}{8}$ **57.** $\frac{1}{16}$ **58.** $\frac{5}{8}$ **59.** $\frac{3}{4}$

60. $\frac{15}{16}$ **61.** $\frac{3}{2}$ **62.** $\frac{5}{4}$ **63.** $\frac{31}{16}$ **64.** $\frac{15}{8}$



65. Write each fraction as an equivalent fraction with denominator 100. Then write them in order from smallest to largest.

$$\frac{3}{10} \quad \frac{1}{20} \quad \frac{4}{25} \quad \frac{2}{5}$$

66. Write each fraction as an equivalent fraction with denominator 30. Then write them in order from smallest to largest.

$$\frac{1}{15} \quad \frac{5}{6} \quad \frac{7}{10} \quad \frac{1}{2}$$

Applying the Concepts

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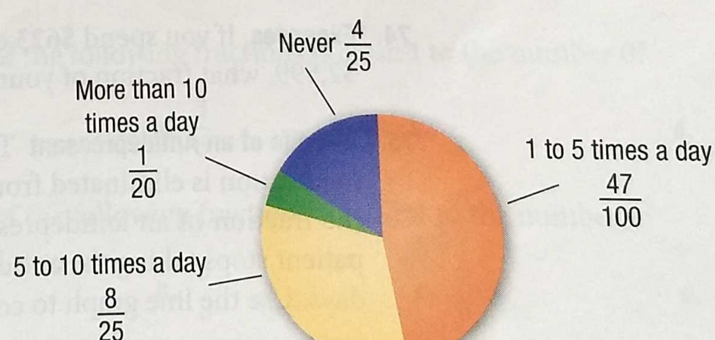
67. Sending E-mail The pie chart below shows the fraction of workers who responded to a survey about sending non-work-related e-mail from the office. Use the pie chart to fill in the table.

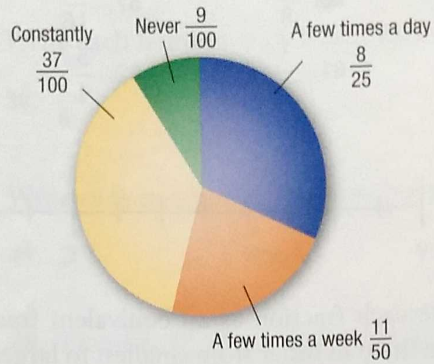
Workers sending personal e-mail from the office



How Often Workers Send Non-Work-Related e-Mail from the Office	Fraction of Respondents Saying Yes
never	
1 to 5 times a day	
5 to 10 times a day	
more than 10 times a day	

68. Surfing the Internet The pie chart below shows the fraction of workers who responded to a survey about viewing non-work-related sites during working hours. Use the pie chart to fill in the table.

Workers surfing the net from the office



How Often Workers View Non-Work-Related Sites from the Office	Fraction of Respondents Saying Yes
never	
a few times a week	
a few times a day	
constantly	

69. Number of Children If there are 3 girls in a family with 5 children, then we say that $\frac{3}{5}$ of the children are girls. If there are 4 girls in a family with 5 children, what fraction of the children are girls?

70. Medical School If 3 out of every 7 people who apply to medical school actually get accepted, what fraction of the people who apply get accepted?

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71. Number of Students Of the 43 people who started a math class meeting at 10:00 each morning, only 29 finished the class. What fraction of the people finished the class?

72. Number of Students In a class of 51 students, 23 are freshmen and 28 are juniors. What fraction of the students are freshmen?

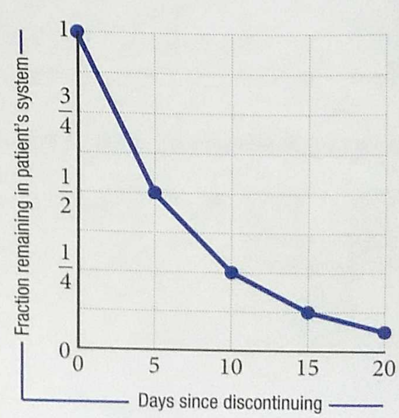
73. Expenses If your monthly income is $1,791 and your house payment is $1,121, what fraction of your monthly income must go to pay your house payment?

74. Expenses If you spend $623 on food each month and your monthly income is $2,599, what fraction of your monthly income do you spend on food?

75. Half-life of an Antidepressant The half-life of a medication tells how quickly the medication is eliminated from a person's system. The line graph below shows the fraction of an antidepressant that remains in a patient's system once the patient stops taking the antidepressant. The half-life of the antidepressant is 5 days. Use the line graph to complete the table.

Concentration of Antidepressant

Days Since Discontinuing	Fraction Remaining in Patient's System
0	1
5	
	$\frac{1}{4}$
15	
	$\frac{1}{16}$



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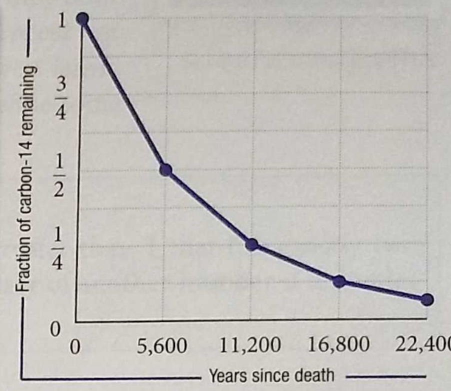
76. Carbon Dating All living things contain a small amount of carbon-14, which is radioactive and decays. The half-life of carbon-14 is 5,600 years. During the lifetime of an organism, the carbon-14 is replenished, but after its death the carbon-14 begins to disappear. By measuring the amount left, the age of the organism can be determined with surprising accuracy. The line graph below shows the fraction of carbon-14 remaining after the death of an organism. Use the line graph to complete the table.

Concentration of Carbon-14

Years Since Death of Organism	Fraction of Carbon-14 Remaining
0	1
5,600	
	$\frac{1}{4}$
16,800	
	$\frac{1}{16}$



Estimating

77. Which of the following fractions is closest to the number 0?

a. $\frac{1}{2}$ **b.** $\frac{1}{3}$ **c.** $\frac{1}{4}$ **d.** $\frac{1}{5}$

78. Which of the following fractions is closest to the number 1?

a. $\frac{1}{2}$ **b.** $\frac{1}{3}$ **c.** $\frac{1}{4}$ **d.** $\frac{1}{5}$

79. Which of the following fractions is closest to the number 0?

a. $\frac{1}{8}$ **b.** $\frac{3}{8}$ **c.** $\frac{5}{8}$ **d.** $\frac{7}{8}$

80. Which of the following fractions is closest to the number 1?

a. $\frac{1}{8}$ **b.** $\frac{3}{8}$ **c.** $\frac{5}{8}$ **d.** $\frac{7}{8}$

Getting Ready for the Next Section

Multiply.

81. $2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$ **82.** $2^2 \cdot 3^3$ **83.** $2^2 \cdot 3 \cdot 5$ **84.** $2^2 \cdot 5$

Divide.

85. $12 \div 3$ **86.** $15 \div 3$ **87.** $20 \div 4$ **88.** $24 \div 4$

89. $42 \div 6$ **90.** $6 \div 6$ **91.** $40 \div 4$ **92.** $105 \div 15$